

## Phys. 611: Homework II

due January 19, 2005

### 1. Commutation Relations (2 pts each)

Consider the relations between commutators of operators and their exponentials. Prove that if  $[\mathbf{P}, \mathbf{Q}] = \lambda \mathbf{1}$ , then

(a)  $e^{\mathbf{P}+\mathbf{Q}} = e^{\mathbf{P}} e^{\mathbf{Q}} e^{-\frac{\lambda}{2}\mathbf{1}}$

(Hint: Consider operators  $E(\mathbf{R}) = e^{\alpha\mathbf{R}}$  and show that both sides of (a) satisfy the same first order differential equation with respect to  $\alpha$ .)

(b)  $e^{\mathbf{P}} e^{\mathbf{Q}} = e^{\mathbf{Q}} e^{\mathbf{P}} e^{\lambda\mathbf{1}}$ .

### 2. Normalized Eigenfunctions (3 pts total)

Let  $\phi_1$  and  $\phi_2$  be the normalized eigenfunctions corresponding to the same eigenvalue. If

$$\int \phi_1^* \phi_2 d\tau = d,$$

where  $d$  is real, find normalized linear combinations of  $\phi_1$  and  $\phi_2$  that are orthogonal to

(a)  $\phi_1$

(b)  $\phi_1 + \phi_2$ .

### 3. Parity (2 pts each)

Modern physics emphasizes the property of parity, i.e. whether a quantity remains invariant or changes sign under an inversion of the coordinate system. In Cartesian coordinates this means  $(x, y, z) \rightarrow (-x, -y, -z)$ .

(a) Show algebraically **and** geometrically that the inversion (reflection through the origin) of a point  $(r, \theta, \varphi)$  relative to *fixed*  $x$ -,  $y$ -,  $z$ -axes consists of the transformation

$$\begin{aligned} r &\rightarrow r \\ \theta &\rightarrow \pi - \theta \\ \varphi &\rightarrow \varphi \pm \pi \end{aligned} \tag{1}$$

- (b) Show that the unit vectors  $\hat{\mathbf{r}}$  and  $\hat{\phi}$  have odd parity (reversal of direction) and that  $\hat{\theta}$  has even parity.

**4. Parity Eigenfunctions** (2 pts each) Consider an real valued function  $f(x)$  expanded in terms of even and odd functions

$$f(x) = c_E \tilde{f}_E(x) + c_O \tilde{f}_O(x), \quad (2)$$

where  $\tilde{f}_{E,O}$  are normalized.

- (a) Show that the probability that the wave function is in an even state is given by

$$c_E^2 = \frac{1}{2} \left( 1 + \frac{\int_{-\infty}^{\infty} dx f(x)f(-x)}{\int_{-\infty}^{\infty} dx f^2(x)} \right) \quad (3)$$

and in an odd state

$$c_O^2 = \frac{1}{2} \left( 1 - \frac{\int_{-\infty}^{\infty} dx f(x)f(-x)}{\int_{-\infty}^{\infty} dx f^2(x)} \right) \quad (4)$$

- (b) Show explicitly that  $c_E^2 = 1$  and  $c_O^2 = 0$  for an even function.

**5. Root-Mean-Square Deviation** (3 pts)

Proof that in an eigenstate of  $\mathbf{L}^2$  and  $\mathbf{L}_3$  the root-mean-square deviation of the other two components is non-zero by calculating

$$(\Delta \mathbf{L}_1)_{l,m} \quad \text{and} \quad (\Delta \mathbf{L}_2)_{l,m} \quad (5)$$

*Hint:*

$$\mathbf{L}_1 = \frac{1}{2}(\mathbf{L}_+ + \mathbf{L}_-) \quad , \quad \mathbf{L}_2 = \frac{1}{i2}(\mathbf{L}_+ - \mathbf{L}_-)$$