Phys. 611: Homework VII

due February 23, 2004

1. Rotation Matrices

(a) (5 pts)
Find the explicit matrix elements of the rotation operator for states having $j = 1/2$ directly from the definition

$$D^j_{mm'}(\alpha, \beta, \gamma) = \langle jm|D(\alpha, \beta, \gamma)|jm'\rangle = e^{-i\alpha m}\, d^j_{mm'}\, e^{-i\gamma m'}, \tag{1}$$

with

$$d^j_{mm'} = \langle jm|e^{-i\beta J_y}|jm'\rangle. \tag{2}$$

Hints: Define the matrix

$$i\beta J_y = A = \frac{\beta}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3}$$

and compute the powers A^{2n} and A^{2n+1}, where n is an integer. Now compare with the power series expansion of the above exponential operator, rearrange the series and collapse then into sin and cos functions. Calculate the matrix elements for all possible values of m and m'.

(b) (3 pts)
Show that

$$D^{j\dagger}(\alpha, \beta, \gamma) = D^j(-\gamma, -\beta, -\alpha). \tag{4}$$

Explain, why this should be.

2. Pauli Matrices I (*2 p each*)

The three Pauli matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5}$$

Show that

(a) $\sigma_i^2 = 1$

(b) $\sigma_i \sigma_j = i\sigma_k$ for (i,j,k) being cyclic permutations

(c) $\sigma_i\sigma_j + \sigma_j\sigma_i = [\sigma_i, \sigma_j]_+ = 2\delta_{ij}1$

3. Pauli Matrices II *(3 p)*

Prove the identity

$$(\vec{\sigma} \cdot \vec{A})\,(\vec{\sigma} \cdot \vec{B}) = \vec{A} \cdot \vec{B} + i\vec{\sigma} \cdot (\vec{A} \times \vec{B}), \tag{6}$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices, a nd $\vec{A}$ and $\vec{B}$ are vector operators that commute with $\vec{\sigma}$, but not necessarily commute with each other.

4. Pauli Matrices III *(3 p)*

Consider an arbitrary matrix

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}. \tag{7}$$

Show that the matrix M can always be written as a linear combination of the four matrices $\mathbf{1}, \sigma_x, \sigma_y, \sigma_z$ as

$$M = a_0 \mathbf{1} + \vec{a} \cdot \vec{\sigma}. \tag{8}$$

where a_0, a_x, a_y, and a_z are complex numbers

5. Schwinger Method for Angular Momentum *(5 p)*

Let $a_1, a_1^\dagger$ and $a_2, a_2^\dagger$ be simple harmonic oscillator annihilation and creation operators. Define

$$\begin{aligned} J_1 &= \frac{\hbar}{2}\left(a_2^\dagger a_1 + a_1^\dagger a_2\right) \\ J_2 &= \frac{i\hbar}{2}\left(a_2^\dagger a_1 - a_1^\dagger a_2\right) \\ J_3 &= \frac{\hbar}{2}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right) \\ j &= \frac{\hbar}{2}\left(a_1^\dagger a_1 + a_2^\dagger a_2\right) \qquad (9) \\ & \qquad\qquad (10) \end{aligned}$$

Show that J_1, J_2, J_3 obey the angular momentum commutation relations

$$[J_1, J_2] = i\hbar J_3$$

and cyclic permutations, and that

$$[j, \vec{J}] = 0$$

as well as that

$$\vec{J}^2 = j(j+1)\hbar^2$$