Phys. 735: Homework II

due September 25, 2002

1. *(10 pts)*
For the simple harmonic oscillator the wave function of an energy eigenstate is given by

$$u_n(x)\exp\left(-\frac{i}{\hbar}E_n t\right) = c_n \exp\left(-\frac{1}{2}\alpha^2 x^2\right) H_n(\alpha x),$$

where

$$\alpha = \sqrt{\frac{m\omega}{\hbar}}\ ;\ \ c_n^2 = \frac{1}{2^n\, n!}\frac{\alpha}{\sqrt{\pi}}\ ;\ \ E_n = \hbar\omega\left(n+\frac{1}{2}\right)$$

Determine the propagator $\mathcal{K}(x'', t; x', t_0)$, where you may consider $t_0 = 0$ for simplicity. Discuss the time behavior of the wave function.
Potentially useful formulae:

$$\exp\left(-\lambda^2 + 2\lambda\eta\right) = \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}H_n(\eta)$$

$$\left(\frac{1}{\sqrt{1-\gamma^2}}\right)\exp\left(\frac{-(\alpha^2+\beta^2-2\alpha\beta\gamma)}{(1-\gamma^2)}\right) = \exp[-(\alpha^2+\beta^2)]\sum_{n=0}^{\infty}\left(\frac{\gamma^n}{2^n n!}\right)H_n(\alpha)H_n(\beta)$$

2. Galilei Invariance of the free Schrödinger Equation *(10 pts)*
Derive the transformation law for the wave function $\Psi(\vec{x}, t)$ of a free particle under a Galilei transformation

$$\vec{x}' = \vec{x} + \vec{v}t + \vec{a}, \quad t' = t + b,$$

from the constraint that the Schrödinger equation has the same form in both systems of inertia

$$\Psi = \Psi(\vec{x}, t):\ \ i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi$$

$$\Psi' = \Psi'(\vec{x}', t'):\ \ i\hbar\frac{\partial\Psi'}{\partial t'} = -\frac{\hbar^2}{2m}\Delta'\Psi'$$

How does the probability current $\vec{j} = \frac{\hbar}{2mi}(\Psi^*\vec{\Delta}\Psi - (\vec{\Delta}\Psi^*)\Psi)$ transform?

Hints: There should be the following relation $\Psi'(\vec{x}', t) = e^{if(\vec{x},t)}\Psi(\vec{x}, t)$ between the wave functions Ψ and Ψ' where f is a real function (why?). Assume that f is independent from Ψ and that the derivatives exist. Derive an equation for f from the above relation between the two wave functions and by using the Schrödinger equation.