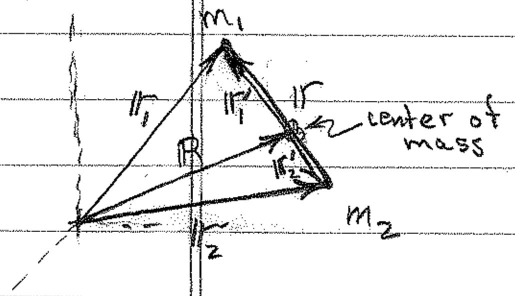
Added note on the change of coords. from $\mathbf{r}_1, \mathbf{r}_2$ to $\mathbf{R}$ and $\mathbf{r}$.



$\mathbf{r}_1 \equiv \mathbf{r}_2 + \mathbf{r}$ — definition of $\mathbf{r}$

$M \cdot \mathbf{R} \equiv m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2$ — definition of center of mass (c.m.)

$M \equiv m_1 + m_2$

$$\left.\begin{aligned} \mathbf{r}_1 &= \mathbf{R} + \mathbf{r}_1' \\ \mathbf{r}_2 &= \mathbf{R} + \mathbf{r}_2' \end{aligned}\right\}$$ Definition of $\mathbf{r}_1'$ & $\mathbf{r}_2'$ giving the position of m_1 & m_2 w.r.t. the c.m.

Note: $m_1 \mathbf{r}_1' + m_2 \mathbf{r}_2' = 0$

Thus, we can define μ such that $\boxed{m_1 \mathbf{r}_1' = \mu \mathbf{r}}$

Then $m_2 \mathbf{r}_2' = -\mu \mathbf{r}$

μ is called "the reduced mass" or "the effective mass".

And since $\mathbf{r} = \mathbf{r}_1' - \mathbf{r}_2'$

we have $\mathbf{r} = \mu\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\mathbf{r} \Rightarrow \boxed{\frac{1}{\mu} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)}$

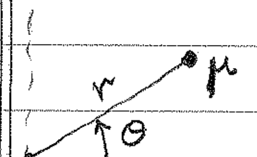
Important note on the physical interpretation of results:

We show that the general two-body problem reduces to the motion of the center of mass $\mathbf{R}(t)$ — easy to solve

Plus the motion of the effective mass μ in a at position $\mathbf{r}$ moving in central potential $V(r)$.

This motion reduces to motion in a plane.

So the solution is for $r(t)$, $\theta(t)$.



Physically we want $\mathbf{r}_1'$ & $\mathbf{r}_2'$ for the motion of m_1 & m_2 relative to the C.M.

But since $\mathbf{r}_1' = \frac{\mu}{m_1}\mathbf{r}$, (& $\mathbf{r}_2' = -\frac{\mu}{m_2}\mathbf{r}$)

the motion of m_1 r.t. c.m is just given by $r' = \frac{\mu}{m_1} r$, $\theta' = \theta$, i.e. only a scale factor μ/m_1.

Summary of properties for orbits in central potential $V(r) = -k/r$ $\quad k \begin{cases} >0 & \text{attractive} \\ <0 & \text{repulsive} \end{cases}$

Orbit eq. gives general orbit:

$$r(\theta) = \frac{C}{1+\epsilon\cos\theta}, \quad C = \frac{\ell^2}{\mu k}, \quad \epsilon^2 = 1 + \frac{2\ell^2 E}{\mu k^2}$$

or $E = \frac{\mu k^2}{2\ell^2}(\epsilon^2 - 1)$

$\epsilon \equiv$ eccentricity

$r_1 \equiv$ closest approach $= \frac{C}{1+\epsilon}$ $\qquad \mu = \frac{m_1 m_2}{m_1+m_2} > 0$.

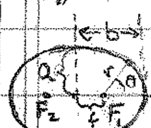
For $k>0$ (attractive force) $\Rightarrow C > 0$

E always $\geq -\frac{\mu k^2}{2\ell^2}$ $\equiv$ energy of circular orbit $\epsilon = 0$.

$r_{circ} = C = \frac{\ell^2}{\mu k}$

Bound orbits:

$k>0$ $\qquad -\frac{\mu k^2}{2\ell^2} \leq E < 0 \iff 0 \leq \epsilon < 1 \iff$ Bound orbits



Bound orbits are ellipses: semimajor axis $a = C/(1-\epsilon^2)$; semiminor axis $b = C/(1-\epsilon^2)^{1/2}$

$f = \epsilon a$ $\qquad r_1 \equiv r_p =$ periapsis

$E = -k/2a \qquad a = k/2|E|$

$b = \ell/\sqrt{2\mu E}$

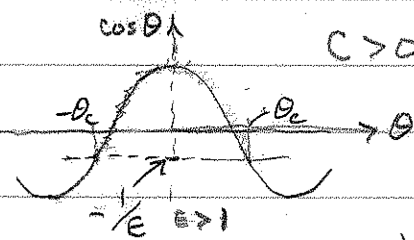
$k>0$ attractive force

Unbound orbits:

$E = 0 \iff \epsilon = 1 \qquad (E=0 \Rightarrow \dot{r}_\infty = \dot{r}_\infty = 0)$

$r(\theta) = \frac{1}{1+\cos\theta}$ orbit is a parabola

$E > 0 \Leftrightarrow |\epsilon| > 1$ together with $\left.\begin{matrix} k>0 \\ r = r_1 \text{ at } \theta = 0 \end{matrix}\right\} \Rightarrow \epsilon > +1$



$C > 0 \Rightarrow 1 + \epsilon\cos\theta \geq 0 \Rightarrow -\theta_c \leq \theta \leq \theta_c$

$\cos\theta_c = -1/\epsilon$

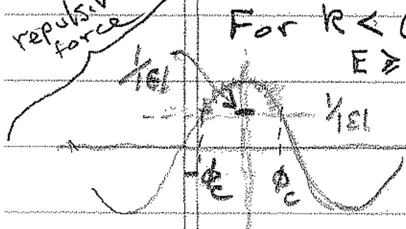
$r \to \infty$ at $\pm\theta_c$, $|\theta_c| > \frac{\pi}{2}$

orbit is a hyperbola (attractive force hyperbola)

repulsive force

For $k < 0$ (repulsive force) $C < 0$ $\quad r = r_1$ at $\theta = 0 \Rightarrow \epsilon < -1$

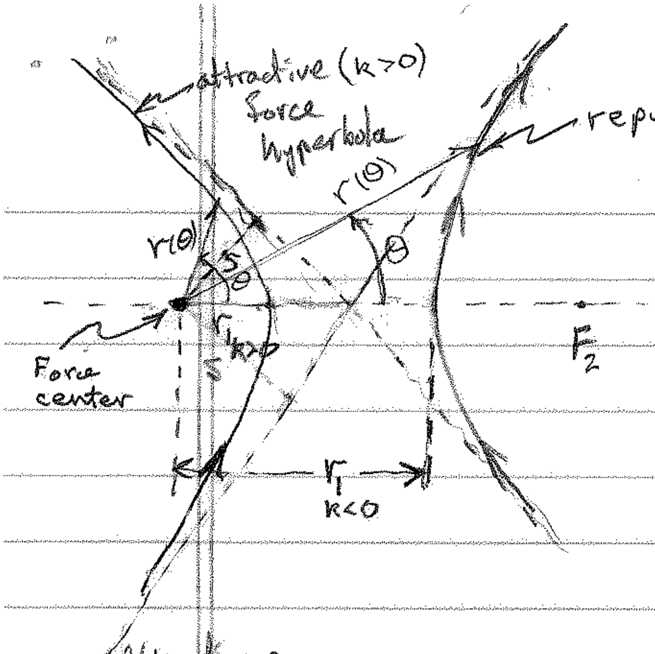
$E \geq 0$ always.



$1 - |\epsilon|\cos\theta \leq 0 \Rightarrow -\phi_c < \theta < \phi_c$

$1 \leq |\epsilon|\cos\theta$ $\qquad \cos\phi_c \geq \frac{1}{|\epsilon|}, \quad |\phi_c| < \frac{\pi}{2}$

orbit is a hyperbola (repulsive force hyperbola)



$V(r) = -k/r$.
For a given $k \neq 0$ and $E > 0$ will get a hyperbolic orbit. Either the attractive $(k>0)$ or the repulsive $(k<0)$ hyperbola.

$$\cos\theta_c = -\frac{1}{\varepsilon} \qquad \varepsilon > 1$$

$$\cos\phi_c = \frac{1}{|\varepsilon|} \qquad |\varepsilon| > 1 \qquad \varepsilon < -1$$

Attractive:

θ goes from $-\theta_c$ to $+\theta_c$ giving $+l > 0$

repulsive:

θ goes from $-\phi_c$ to $+\phi_c$ giving $+l > 0$

Impact parameter: $s \Rightarrow l = \mu v_\infty \cdot s = \mu v_\infty s$

$$E = \tfrac{1}{2}\mu v_\infty^2 = \tfrac{1}{2}\mu v_\infty^2$$ because $V = 0$ when $r = \infty$.

or $v_\infty = \sqrt{\frac{2E}{\mu}}$

$\therefore$ $$s = l/\sqrt{2\mu E} = \frac{1}{2E}\sqrt{\frac{2}{\mu}}\, l\sqrt{E}$$

Since $E = \frac{\mu k^2}{2l^2}(\epsilon^2 - 1)$

$$s = \frac{|k|}{2E}\sqrt{\epsilon^2-1} \quad \text{or} \quad s = \frac{l^2}{\mu |k| \sqrt{\epsilon^2 - 1}}$$

$$r_{1\,k>0} = \frac{C}{1+\epsilon}$$

$$r_{1\,k<0} = \frac{C}{1+\epsilon} = \frac{-|C|}{1-|\epsilon|} = \frac{|C|}{|\epsilon|-1}$$

Suppose l, E, $|k|$ same in both cases. $|C|$ and $|\epsilon|$ are the same.

But the distance of closest approach will <u>not</u> be the same in the two cases.

$$r_{1\,k>0} = \frac{|C|}{1+|\epsilon|} < r_{1\,k<0} = \frac{|C|}{|\epsilon|-1} \qquad |\epsilon| > 1$$