w. solns

KUK

Phys 605. Homework 2

Due 5pm, Monday, September 22, 2008

Problem 2-1: [10 pts.] Consider a system of interacting particles, each with mass m_i. Show that the magnitude R_{CM} of the position vector for the center of mass is given by

$$M^2 R_{CM}^2 = M \sum_i m_i r_i^2 - \frac{1}{2} \sum_{i,j} m_i m_j r_{ij}^2,$$

where r_i is the magnitude of the position vector of the i-th particle and r_{ij} is the distance between particles i and j. Note: take $r_{ii} \equiv 0$.

Problem 2-2: [10 pts.] Consider a thin vertical disk of radius a rolling without slipping on a horizontal plane (see pg. 15 in the Goldstein text and/or my class notes Section 2, pg. 2-6). Four generalized coordinates describing the configuration of the disk at any time t (as defined in the class notes and text) are (x, y, θ, ϕ). The discussion in the text and class notes shows that the rolling without friction constraint is expressed by two differential constraint equations:

$$dx - a \sin\theta d\phi = 0,$$
$$dy + a \cos\theta d\phi = 0.$$

This problem is to show that this constraint is nonholonomic. If it was a holonomic constraint, then two functions would exist such that $G_1(x, y, \theta, \phi) = 0$ and $G_2(x, y, \theta, \phi) = 0$ and the differential constraint equations above would be the exact differentials: $dG_1 = 0$ and $dG_2 = 0$. Thus, we need to show that it is impossible to find two such functions.

(a) Show that neither of these two differential constraint equations is an exact differential as written. Hint: see class notes pg 2-7a.

(b) Further, show that no integrating factor function exists (see class notes pg 2-7b) for either differential constraint equation that will turn it into an exact differential.

Completing (b) we have shown neither differential constraint equation could be an exact differential. All we really needed to do was show one of the two differential constraint equations could not be an exact differential to show that a disk rolling without slipping on a plane is a nonholonomic constraint.

Problem 2-3: [20 pts.] Two point masses, each of mass m, are joined by a rigid weightless rod of length ℓ, the center of which is constrained to remain on a circle of radius a. The circle of radius a lies in a vertical plane. [Only the center of the rod is constrained to the plane of the circle; the masses at the ends of the rod are thus constrained to a sphere (of radius $\ell/2$) and the center of the sphere remains on the circle.]

(a) Define a set of generalized coordinates for this system.

(b) Express the total kinetic energy of the masses in terms of your generalized coordinates.

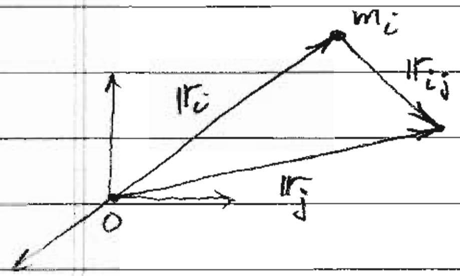
(c) Assume the whole system is in a uniform gravitational field g directed downward. Write an expression for the potential energy of the system.

(d) Find the Lagrange equations of motion for this system.

PWR

Clarified Goldstein, Poole, and Safko, 3rd ed. Prob. 2 Ch. 1 Solution:

Show that $M^2 R_{CM}^2 = M \sum_i m_i r_i^2 - \frac{1}{2} \sum_{i,j} m_i m_j r_{ij}^2$.



Definitions: $\mathbf{r}_j = \mathbf{r}_i + \mathbf{r}_{ij} \Rightarrow \mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i$

$M \equiv \sum_i m_i$ Note $r_{ii} = 0$ for all i.

$M \mathbf{R}_{CM} = \sum_i m_i \mathbf{r}_i$

Proof: $M^2 R_{CM}^2 = \sum_i m_i \mathbf{r}_i \cdot \sum_j m_j \mathbf{r}_j = \sum_{i,j} m_i m_j \mathbf{r}_i \cdot \mathbf{r}_j$

$$M^2 R_{CM}^2 = \sum_{i,j} m_i m_j \mathbf{r}_i \cdot (\mathbf{r}_i + \mathbf{r}_{ij}) = \sum_{ij} m_j m_i r_i^2 + \sum_{ij} m_i m_j \mathbf{r}_i \cdot \mathbf{r}_{ij}$$

$$M^2 R_{CM}^2 = M \sum_i m_i r_i^2 + \sum_{ij} m_i m_j \mathbf{r}_i \cdot \mathbf{r}_{ij} \qquad (1)$$

To complete the proof we must show that

$\sum_{ij} m_i m_j \mathbf{r}_i \cdot \mathbf{r}_{ij}$ is equal to $-\frac{1}{2} \sum_{ij} m_i m_j r_{ij}^2$

Start with $\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i$.

$$r_{ij}^2 \equiv \mathbf{r}_{ij} \cdot \mathbf{r}_{ij} = (\mathbf{r}_j - \mathbf{r}_i) \cdot (\mathbf{r}_j - \mathbf{r}_i) = r_j^2 + r_i^2 - 2 \mathbf{r}_i \cdot \mathbf{r}_j.$$

Thus $\sum_{ij} m_i m_j r_{ij}^2 = \sum_{ij} m_i m_j r_j^2 + \sum_{ij} m_i m_j r_i^2 - 2 \sum_{ij} m_i m_j \mathbf{r}_i \cdot \mathbf{r}_j$

or $\sum_{ij} m_i m_j r_{ij}^2 = 2M \sum_i m_i r_i^2 - 2 \sum_{ij} m_i m_j \mathbf{r}_i \cdot (\mathbf{r}_i + \mathbf{r}_{ij})$

$$\sum_{ij} m_i m_j r_{ij}^2 = \cancel{2M \sum_i m_i r_i^2} - \cancel{2M \sum_i m_i r_i^2} - 2 \sum_{ij} m_i m_j \mathbf{r}_i \cdot \mathbf{r}_{ij}$$

Thus, $\sum_{ij} m_i m_j \mathbf{r}_i \cdot \mathbf{r}_{ij} = -\frac{1}{2} \sum_{ij} m_i m_j r_{ij}^2 \qquad (2)$

Hence, eqns (1) & (2) provide the desired result. Done!

Show disk rolling w/o slipping on a plane is nonholonomic constraint soln: PWR

Start from class notes pg 2-6:

For vertical disk rolling w/o slipping on a horizontal plane:

Generalized coords. $\{q\} = (x, y, \theta, \phi)$

Thus the most general differential constraint eqn. will be of the form:

$$g_x\,dx + g_y\,dy + g_\theta\,d\theta + g_\phi\,d\phi = 0$$

The first constraint eq. for disk rolling w/o slipping on plane is

Eq. (1) $$dx - a\sin\theta\,d\phi = 0$$

$\therefore$ for this constraint eqn: $g_x = 1$, $g_y = 0$, $g_\theta = 0$, and $g_\phi = -a\sin\theta$

If Eq. (1) is an exact differential then there exists a function $G(x,y,\theta,\phi) = 0$ such that Eq. (1) is given by $dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial y}dy + \frac{\partial G}{\partial \theta}d\theta + \frac{\partial G}{\partial \phi}d\phi = 0$

$\therefore$ $\frac{\partial G}{\partial x} = g_x$, $\frac{\partial G}{\partial y} = g_y$, $\frac{\partial G}{\partial \theta} = g_\theta$, $\frac{\partial G}{\partial \phi} = g_\phi$ and

$$\frac{\partial^2 G}{\partial y \partial x} = \frac{\partial^2 G}{\partial x \partial y} \Rightarrow \frac{\partial g_x}{\partial y} = \frac{\partial g_y}{\partial x} \quad \text{etc.}$$

(a) Note that, <u>as it stands</u>, Eq. (1) is not an exact differential since

$$\frac{\partial g_\phi}{\partial \theta} = -a\cos\theta \neq \frac{\partial g_\theta}{\partial \phi} = 0.$$

(b) Can we find an integrating function $f(x,y,\theta,\phi)$ s.t.

$f\,dx - f\,a\sin\theta\,d\phi = 0$ is an exact differential?

Check the mixed partial derivatives:

$$\frac{\partial (f g_y)}{\partial x} = \frac{\partial (f g_x)}{\partial y} \Rightarrow 0 = \frac{\partial f}{\partial y} \Rightarrow f \text{ cannot depend on } y$$

$\therefore$ $f(x, \theta, \phi)$

Disk rolling w/o slipping on plane – nonholonomic constraint
soln continued:

$f(x,\theta,\phi)$

$$\frac{\partial(f g_\theta)}{\partial x} = \frac{\partial(f g_x)}{\partial \theta} \Rightarrow 0 = \frac{\partial f}{\partial \theta} \Rightarrow f \text{ cannot depend on } \theta$$

$\therefore f(x,\phi)$

$$\frac{\partial(f g_\phi)}{\partial x} = \frac{\partial(f g_x)}{\partial \phi} \Rightarrow -a\sin\theta \frac{\partial f(x,\phi)}{\partial x} = \frac{\partial f(x,\phi)}{\partial \phi}$$

Take $\frac{\partial}{\partial\theta}$ of both sides $\Rightarrow -a\cos\theta \frac{\partial f(x,\phi)}{\partial x} = 0$

$$\Rightarrow \frac{\partial f}{\partial x} = 0$$

and then $\frac{\partial f}{\partial \phi} = 0$

$\therefore$ f cannot depend on any variable or f = constant. But Eq. ① is not an exact differential as it stands, and multiplying by any constant will not change that.

Thus, no integrating function exists that will turn Eq. ① into an exact differential and Eq. ① is a non-holonomic constraint. Done for first constraint Eqn.

→ Must do same thing for the second constraint eqn. – find that no integrating function exist for it either.

Note added to solution:

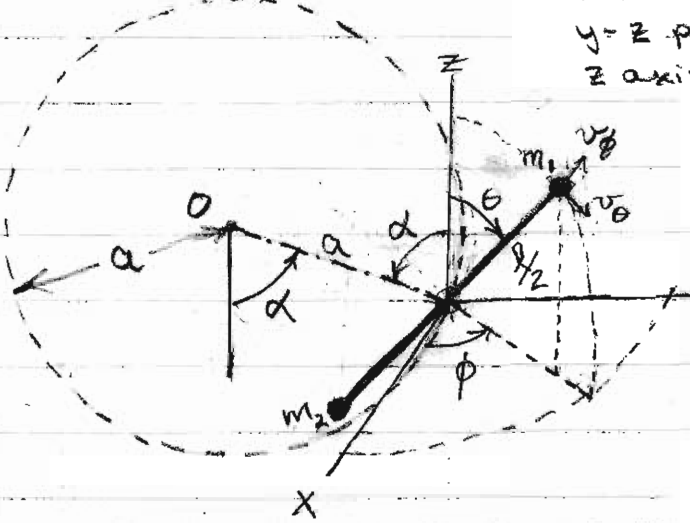
If we had chosen $\frac{\partial(f(x,\phi) g_\theta)}{\partial \phi} = \frac{\partial(f(x,\phi) g_\phi)}{\partial \theta} \Rightarrow 0 = -f(x,\phi)\, a\cos\theta$

$\Rightarrow f(x,\phi) = 0$ which means not only must f = const but that the const. = 0!

Dumbell masses w. CM constrained to verticle circle prob. — soln:

(a)

Origin of xyz coord sys, is at center of rod.
Orientation of x, y, z axes fixed in space
y-z plane is in the vertical plane of circle
z axis is in upward vertical direction.



Use spherical polar coord. relative to x, y, z axes to give positions of the two masses relative to the CM which is at the center of the rod and constrained to the vertical circle of radius a.

mass $m_1 = m_2 = m$ at the ends of light rod (with center at CM) are constrained to move on surface of sphere of radius $\ell/2$ centered on CoM.

Generalized Coords. (α, θ, ϕ) (all angles)
Want to find Total KE $T(\alpha, \dot\alpha, \theta, \dot\theta, \phi, \dot\phi)$.

(b)

$T =$ Total KE = KE of CM + KE rel. to CM.

$$T = T_{CM} + T_{\text{relative to CM}}$$

$$T_{CM} = \frac{1}{2}\cdot 2m(a\dot\alpha)^2 = ma^2\dot\alpha^2$$

Because connected by rigid rod with center of mass at its center, the velocity of mass 1 relative to the center of mass $\equiv \vec{v}_1{}' = -\vec{v}_2{}'$

$$\therefore \quad \frac{1}{2}m v_1'^2 = \frac{1}{2}m v_2'^2$$

$T_{\text{rel CM}}$:

Use spherical coord with center at C.M.

$$\mathbf{v}' = v_r\hat{\mathbf{r}} + v_\theta\hat{\theta} + v_\phi\hat{\phi}$$

$$v_r = 0 \;;\quad v_\theta = \frac{\ell}{2}\dot\theta \;;\quad v_\phi = \frac{\ell}{2}\sin\theta\,\dot\phi$$

$$T_{\text{rel CM}} = \frac{2m}{2}\left(v_\theta^2 + v_\phi^2\right) = m\frac{\ell^2}{4}\left(\dot\theta^2 + \sin^2\theta\,\dot\phi^2\right)$$

Thus

$$\boxed{T = m\left[a^2\dot\alpha^2 + \frac{\ell^2}{4}\left(\dot\theta^2 + \sin^2\theta\,\dot\phi^2\right)\right]}$$

Dumbell masses prob — soln continued

(c) Potential Energy (meas. relative to the center of the vertical circle of radius a)

is given by

$$V = mgh_1 + mgh_2$$

where

$$h_1 = -a\cos\alpha + \frac{l}{2}\cos\theta$$

$$h_2 = -a\cos\alpha - \frac{l}{2}\cos\theta$$

Thus, $\boxed{V = -2mga\cos\alpha}$

(d) Find the Lagrange Eq. of motions.

Lagrangian $L(\alpha, \theta, \phi, \dot\alpha, \dot\theta, \dot\phi) = T - V$

$$= m\left\{\left[a^2\dot\alpha^2 + \frac{l}{4}(\dot\theta^2 + \sin^2\theta\,\dot\phi^2)\right] + 2ag\cos\alpha\right\}$$

α:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\alpha}\right) - \frac{\partial L}{\partial\alpha} = 0.$$

$$\frac{\partial L}{\partial\dot\alpha} = 2ma^2\dot\alpha \quad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\alpha}\right) = 2ma^2\ddot\alpha \ ; \quad \frac{\partial L}{\partial\alpha} = -2mag\sin\alpha$$

$$\therefore \quad \boxed{a\ddot\alpha + g\sin\alpha = 0}$$

θ:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) - \frac{\partial L}{\partial\theta} = 0. \quad \frac{\partial L}{\partial\dot\theta} = \frac{ml}{2}\dot\theta \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) = \frac{ml}{2}\ddot\theta$$

$$\frac{\partial L}{\partial\theta} = \frac{ml}{2}\dot\phi^2\sin\theta\cos\theta$$

$$\therefore \quad \boxed{\ddot\theta - \dot\phi^2\sin\theta\cos\theta = 0}$$

ϕ:

$$\left.\begin{array}{l}\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\phi}\right) - \frac{\partial L}{\partial\phi} = 0. \\ \partial L/\partial\phi = 0\end{array}\right\} \Rightarrow \frac{\partial L}{\partial\dot\phi} = \text{const} \Rightarrow \boxed{\frac{ml}{2}\sin^2\theta\,\dot\phi = \text{const} = \frac{ml}{2}\sin^2\theta_0\,\dot\phi_0.}$$