Phys 605. Homework 5

Due 5pm, Tuesday, October 14, 2008

Problem 5-1: [20 pts.] A spherical pendulum consists of a particle of mass m in a uniform gravitational field constrained to move on the surface of a sphere of radius R.

(a) Find a Lagrangian for the spherical pendulum Find any constants of the motion.

(b) Show the general motion reduces to a one-dimensional problem for which an effective potential can be defined.

(c) Make a rough sketch of the effective potential and discuss the possible motion.

(d) For proper initial conditions, the mass simply moves in a horizontal circular orbit with $\theta = \theta_0$ and constant angular frequency Ω_0. Show that the angular frequency for small perturbations of the circular orbit is given by

$$\omega^2 = {\Omega_0}^2(1 + 3\cos^2\theta_0).$$

(e) Given the expression for ω in part (d) and assuming $\dot{\phi} \approx \Omega_0$, do the orbits close in general? Briefly provide your reasoning supporting your answer.

Problem 5-2: [10 pts] A particle of mass m follows the orbit $r = a(1 + \cos\theta)$ under the action of a force always directed toward the origin.

(a) Find the force law.

(b) Determine the total energy of the particle in the orbit.

Problem 5-3: [10 pts.] *Goldstein, Poole, and Safko, (3rd. ed.)* Problem 10, Chapter 3, pg. 128.

The following problem was delayed until HW 6.

Problem 5-4: [10 pts.] Consider point particles scattering elastically (angle of incidence equals angle of reflection at point of impact) from an infinitely massive, perfectly hard ellipsoid of rotation. The ellipsoid is obtained by rotating the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ about the x axis. The beam of incident particles is directed along the x-axis.

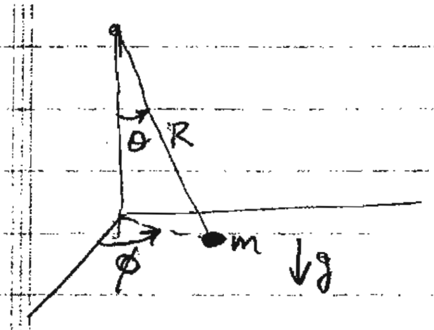
(a) Show that the scattered angle Θ is related to the impact parameter s by

$$s = \frac{b^2}{a}\left[\frac{b^2}{a^2} + \tan^2(\frac{\Theta}{2})\right]^{-1/2}$$

(b) Find the differential scattering cross-section.

(c) What is the total scattering cross-section?

Solution: Spherical Pendulum Problem



(a) $T = \frac{m}{2}R^2\dot{\theta}^2 + \frac{m R^2}{2}\sin^2\theta\,\dot{\phi}^2$

$V = -Rmg\cos\theta + \cancel{\text{const.}}^{\,0}$

Let $I = mR^2$.

$$L = T - V = \frac{I}{2}\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right) + \frac{Ig}{R}\cos\theta$$

Eq. of motion:

ϕ: ϕ is cyclic coord. $\Rightarrow \frac{\partial L}{\partial \dot{\phi}} = \text{const.} \Rightarrow I\sin^2\theta\,\dot{\phi} = p_\phi = \text{const.}$

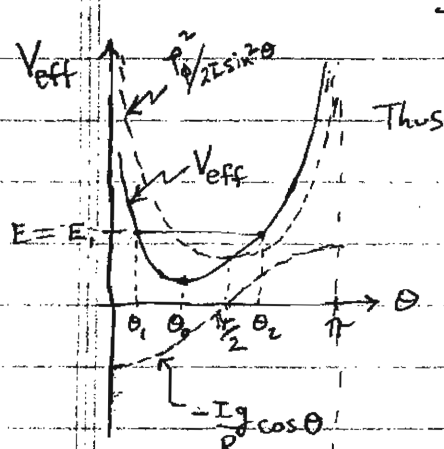
or $\dot{\phi} = p_\phi / I\sin^2\theta$ ①

θ: $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta} \Rightarrow I\ddot{\theta} = I\sin\theta\cos\theta\,\dot{\phi}^2 - \frac{Ig}{R}\sin\theta$

or eliminating $\dot{\phi}$ using ①:

$$I\ddot{\theta} = \underbrace{\frac{p_\phi^2}{I}\frac{\cos\theta}{\sin^3\theta} - \frac{Ig}{R}\sin\theta}_{f(\theta)} \equiv \underbrace{-\frac{dV_{eff}}{d\theta}}_{\text{defines } V_{eff}}$$

(b) Sketch of V_{eff}



Thus

$$\boxed{V_{eff}(\theta) = \frac{p_\phi^2}{2I}\frac{1}{\sin^2\theta} - \frac{Ig}{R}\cos\theta}$$

Since $L(\cancel{t})$, $h = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L = \text{const.}$

$\text{const.} = h = \frac{I}{2}\dot{\theta}^2 + \frac{I}{2}\sin^2\theta\,\dot{\phi}^2 - \frac{Ig}{R}\cos\theta = T + V$

and using ① to eliminate $\dot{\phi}$

$E(\theta,\dot{\theta}) = \frac{I}{2}\dot{\theta}^2 + V_{eff}(\theta) = \text{const.}$

Note: minimum in V_{eff} will always be at $\theta = \theta_0 < \pi/2$

If $p_\phi = 0$ then $\theta_0 = 0$ and have simple pendulum.

∴ General motion will have θ osc. between turning pt. θ_1 and θ_2 as mass moves around z-axis with $\dot{\phi} = p_\phi / I\sin^2\theta$.

Spherical Pendulum Problem – Sol'n cont.: PWR

(c) At the minimum in V_{eff} :

$$\left.\frac{dV_{eff}}{d\theta}\right|_{\theta_0} = -\frac{P_\phi^2 \cos\theta_0}{I \sin^3\theta_0} + \frac{gI}{R}\sin\theta_0 = 0$$

or $$\frac{P_\phi^2}{I}\cos\theta_0 = g\frac{I}{R}\sin^4\theta_0 > 0 \Rightarrow \theta_0 < \pi/2 .$$

For $\theta = \theta_0$ ~~motion~~ orbit: $P_\phi = I\sqrt{\frac{g}{R\cos\theta_0}}\cdot\sin^2\theta_0 \Rightarrow \dot{\phi}_0 \equiv \Omega_0 = \frac{P_\phi}{I\sin^2\theta_0} = \sqrt{\frac{g}{R\cos\theta_0}}$

For small oscillations about θ_0 :

$$\frac{d^2V_{eff}}{d\theta^2} = \frac{P_\phi^2}{I}\left[\frac{1}{\sin^2\theta} + \frac{3\cos^2\theta}{\sin^4\theta}\right] + \frac{gI}{R}\cos\theta$$

At $\theta = \theta_0$

$$\left.\frac{d^2V_{eff}}{d\theta^2}\right|_{\theta_0} = \frac{P_\phi^2}{I}\left[\frac{\sin^2\theta_0 + \cos^2\theta_0 + 2\cos^2\theta_0}{\sin^4\theta_0}\right] + \underbrace{\frac{gI\cos\theta_0}{R}}_{\frac{P_\phi^2}{I}\frac{\cos^2\theta_0}{\sin^4\theta_0}}$$

$$\therefore \left.\frac{d^2V_{eff}}{d\theta^2}\right|_{\theta_0} = \frac{P_\phi^2}{I\sin^4\theta_0}\left[1 + 3\cos^2\theta_0\right] = I\Omega_0^2\left[1 + 3\cos^2\theta_0\right]$$

$\therefore$ Expanded about $\theta = \theta_0$

$$E = \tfrac{1}{2}I\dot{\theta}^2 + V_{eff}(\theta_0) + \tfrac{1}{2}\left[I\Omega_0^2(1+3\cos^2\theta_0)\right](\theta - \theta_0)^2$$

$\therefore$ freq. of small osc. $\Rightarrow$ $\omega^2 = \frac{\cancel{I}\,\Omega_0^2(1+3\cos^2\theta_0)}{\cancel{I}}$.

(d) Orbits will close if $\frac{\omega}{\Omega_0}$ is rational i.e. if $\frac{\omega^2}{\Omega_0^2} = \frac{p^2}{q^2}$ p, q integers.

In our case $\frac{\omega^2}{\Omega^2} = (1 + 3\cos^2\theta_0) = \frac{p^2}{q^2}$ — This will not be satisfied for arbitrary θ_0 .

RWR

5-2 Solution: Sketch

(a) Given: $r = a(1+\cos\theta)$

From orbit equation:

$$u^2 \frac{\ell^2}{m}\left(\frac{d^2u}{d\theta^2} + u\right) = -F(r) \qquad \text{where } u \equiv \frac{1}{r}$$

In this case $u = \dfrac{1}{a(1+\cos\theta)}$

By direct differentiation can show that

$$\frac{d^2u}{d\theta^2} = \frac{1}{a(1+\cos\theta)^3}\left[\sin^2\theta + 1 + \cos\theta\right]$$

and $$\frac{d^2u}{d\theta^2} + u = \frac{3}{a(1+\cos\theta)^2} = \frac{3a}{r^2}$$

Thus $$F(r) = -\frac{3a\ell^2}{m}\frac{1}{r^4}$$

(b) $$-\frac{dV}{dr} = F(r) \Rightarrow V(r) = -\frac{a\ell^2}{m}\frac{1}{r^3} + \cancel{\text{const}}^{\to 0}$$

Note this could be any constant but the convention is to choose the constant so that $V(r) \to 0$ as $r \to \infty$ if that is possible. RWR

$$E = \text{const.} = \frac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mr^2} - \frac{a\ell^2}{m}\frac{1}{r^3}$$

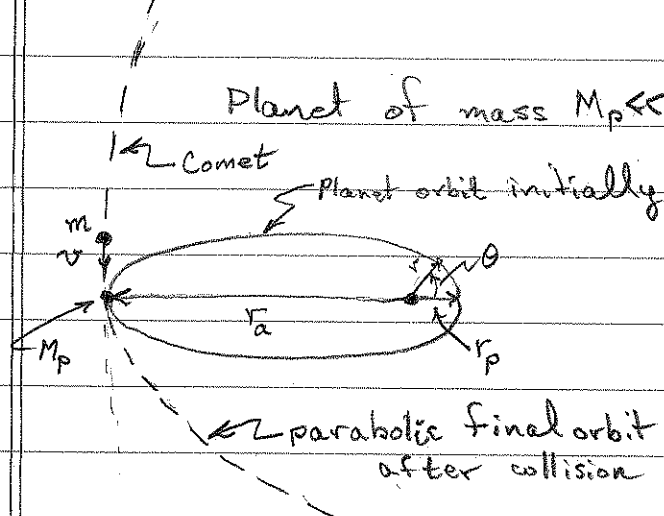
At ~~the~~ outer turning point $r = \text{maximum} = 2a$
and $\dot{r} = 0$

$$\therefore \quad E = \frac{\ell^2}{2m\,4a^2} - \frac{a\ell^2}{m\,8a^3} = 0$$

$$\therefore \quad E = 0 = \text{const.}$$

GPS, Prob. 3-10 Solution:

Planet of mass $M_p \ll M_s$ initially in ellipical orbit about M_s:



Initial Planet orbit:

$$r(\theta) = \frac{C_i}{1+\varepsilon_i\cos\theta} \quad \text{with}$$

$$C_i = \frac{l_i^2}{\mu k}, \quad \varepsilon_i = 1-\alpha \ (\alpha \ll 1)$$

$$\mu = \frac{M_p M_s}{M_p + M_s} \doteq M_p \qquad \varepsilon_i^2 = 1 + \frac{2l_i^2}{\mu k^2}E_i \implies E_i = -\frac{\mu k^2}{l_i^2}\alpha + O(\alpha^2)$$

Before Collision: Planet is at greatest distance from M_s (apoapsis) $\Rightarrow \theta = \pi$

$$r(\theta=\pi) = r_a = \frac{C_i}{1-\varepsilon_i} = \frac{l_i^2/\mu k}{1-1+\alpha} = \frac{l_i^2}{\mu k}\cdot\frac{1}{\alpha}$$

Thus, $\boxed{r_a = \frac{l_i^2}{\mu k}\cdot\frac{1}{\alpha}}$ (1)

The Collision: Completely inelastic. With comet moving tangent to orbit.

T+V is not conserved in the collision but Total Angular momentum is conserved:

$$\boxed{l_f = l_i + r_a m v} \qquad m \ll M_p \qquad (2)$$

l_f: comet stuck to Planet after collision

l_i: Planet before collision

$r_a m v$: angular momentum of comet just before the collision.

After Collision: Comet & Planet move as one body in a parabolic orbit with r_a as the closest approach for the parabola with ang. mom l_f.

GPS Prob 3-10 soln continued

After Collision:

$$r(\theta) = \frac{c_f}{1+\varepsilon_f \cos\theta} \qquad c_f = \frac{l_f^2}{\mu k} \qquad \varepsilon_f = 1 \quad \text{(parabola)}$$

$\mu \approx M_p$ since $m \ll M_p$, i.e., $\mu_f \doteq \mu_i = \mu = M_p$.

Initially for parabola $r(\theta=0) = \underbrace{r_{p_f}}_{\text{periapsis for the parabola}} = \frac{c_f}{1+1} = \frac{c_f}{2} = \underbrace{r_a}_{\text{apoapsis for original ellipse.}}$

Thus $\frac{c_f}{2} = \frac{l_f^2}{2\mu k} = r_a = \frac{l_i^2}{\mu k}\frac{1}{\alpha}$

or $\frac{l_f^2}{2} = \frac{l_i^2}{\alpha}$ or $\boxed{l_f = l_i\sqrt{\frac{2}{\alpha}}}$ (3)

Eq. (3) must be satisfied if final orbit is to be a parabola.

We can now get the minimum velocity, v, of the comet such that we satisfy eq. (3) by using the conservation of angular momentum condition — eq (2).

$$l_f = l_i + r_a m v_{min} = l_i\sqrt{\frac{2}{\alpha}} \qquad \text{where } r_a = \frac{l_i^2}{\mu k}\frac{1}{\alpha} \text{ from (1).}$$

Thus

$$v_{min} = \frac{1}{r_a m} l_i\left(\sqrt{\frac{2}{\alpha}}-1\right) = \frac{\mu k}{m l_i}\alpha\left(\sqrt{\frac{2}{\alpha}}-1\right) = \frac{\mu k}{m l_i}\sqrt{2\alpha}\left(1-\cancel{\sqrt{\frac{\alpha}{2}}}\right).$$

neglect.

Therefore, the KE of the comet must be at least

$$\frac{1}{2} m v_{min}^2 = \frac{\mu}{m}\cdot\frac{\mu k^2}{l_i^2}\alpha \doteq \frac{M_p}{m}\cdot|E_i| \qquad \text{where } \underbrace{E_i = -\frac{\mu k^2}{l_i^2}\alpha}_{\text{Total energy of the planet before the collision.}}$$

GPS Prob. 3-10 Solution continued:

We have found KE of comet must be at least

$$(KE)_{comet} \doteq \frac{M_P}{m} |E_i| \quad \text{where } E_i = \text{total energy of planet ~~before~~ collision.}$$

We needed to add at least $|E_i|$ to the planet's energy to get it into a parabolic orbit (with $E = 0$).

But we had to hit it with a comet with

$$KE_{min} \doteq \frac{M_P}{m} |E_i| \quad \text{where } \frac{M_P}{m} \gg 1 \ !$$

That is because energy transfer in an inelastic collision with $m \ll M_p$ is very inefficient!

It is clear that will get maximum energy transfer (from m to M_p+m) for inelastic collision when $m \gg M_p$.

Get maximum energy transfer in an elastic collision when $m = M$.