with solns
Pew

Phys 605. Homework 9

Due 5pm, Friday, November 14, 2008

Problem 9-1: [10 pts] A Hamiltonian-like formulation of classical mechanics can be obtained by performing a double Legendre transformation from the Lagrangian $L(q, \dot{q}, t)$ to a new "Hamiltonian" $G(p, \dot{p}, t)$ where p and $\dot{p}$ are independent variables.

(a) Show in detail how to construct $G(p, \dot{p}, t)$, and derive the corresponding "Hamilton's equations of motion." that follow from G. [Hint: note that the Lagrangian formulation defines $\partial L/\partial \dot{q}_i \equiv p_i$ and Lagrange's equations can be written as $\partial L/\partial q_i = \dot{p}_i$.]

(b) Demonstrate that your new Hamiltonian-like formulation works by applying it to a one dimensional harmonic oscillator consisting of mass m attached to a spring of spring constant k. Use the new formulation to the equation of motion. [That is: Find the Lagrangian first, then find the function G (from part (a)). Then show that the equations of motion obtained from G lead to the well known equation of motion for the simple harmonic oscillator and that the form of G implies $h = T + V =$ constant of motion.]

Problem 9-2: [20 pts] A particle of mass m moves in three dimensions under the influence of a central force with potential $V(r)$. Using standard spherical polar coordinates (r, θ, ϕ) as generalized coordinates:

(a) Find the Hamiltonian for this system.

(b) Use the Hamiltonian to identify at least two constants of motion.

(c) Write Hamilton's equations of motion for this system.

(d) Show that Hamilton's equations lead to the Lagrange equations obtained from your Lagrangian.

Problem 9-3: [10 pts] Show that the motion in phase space (q,p) specified by the equations

$$\dot{q} = p \sin q + q \cos p$$
$$\dot{p} = -(p \cos q + \sin p),$$

is <u>not</u> described by a Hamiltonian function.

New "Hamiltonian-like formulation" problem — solution

(a) Given the Lagrangian $L(q,\dot{q},t)$ and p defined by

$$\frac{\partial L}{\partial \dot{q}_i} \equiv p_i \quad \text{for all } i.$$

Lagrange's eqs. $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$

written as $\frac{\partial L}{\partial q_i} = \dot{p}_i$ for all i.

Thus, we can perform a double Legendre transformation of $L(q,\dot{q},t)$ replacing q_i with $\dot{p}_i$ (since $\frac{\partial L}{\partial q_i} = \dot{p}_i$) and replacing $\dot{q}_i$ with p_i (since $\frac{\partial L}{\partial \dot{q}_i} = p_i$):

Eq. 1 $$\boxed{G(p,\dot{p},t) = \underbrace{\dot{p}_i q_i}_{\text{sum}} + \underbrace{p_i \dot{q}_i}_{\text{sum}} - L(q,\dot{q},t)}\,.$$

$$dG = \cancel{\dot{p}_i\, dq_i} + q_i\, d\dot{p} + \cancel{p_i\, d\dot{q}_i} + \dot{q}_i\, dp_i - \underbrace{\frac{\partial L}{\partial q_i}}_{\dot{p}_i}\cancel{dq_i} - \underbrace{\frac{\partial L}{\partial \dot{q}_i}}_{p_i}\cancel{d\dot{q}_i} - \frac{\partial L}{\partial t}\,dt$$

also

$$dG = \frac{\partial G}{\partial p_i}\, dp_i + \frac{\partial G}{\partial \dot{p}_i}\, d\dot{p}_i + \frac{\partial G}{\partial t}\, dt$$

The two expressions above must be equal for all $\{dp, d\dot{p}, dt\}$.

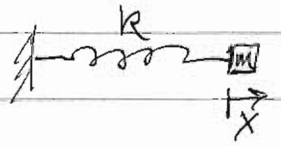
Thus, we have the eqs:

$$\underbrace{\boxed{\frac{\partial G}{\partial p_i} = \dot{q}_i}}_{\text{Eq. 2}}, \quad \underbrace{\boxed{\frac{\partial G}{\partial \dot{p}_i} = q_i}}_{\text{Eq. 3}}, \quad \text{and} \quad \underbrace{\boxed{\frac{\partial G}{\partial t} = -\frac{\partial L}{\partial t}}}_{\text{Eq. 4}} \quad \text{for all } i.$$

From Eq. 4, we see that if $\frac{\partial G}{\partial t} = 0$ then $\frac{\partial L}{\partial t} = 0 \Rightarrow h \equiv p_i \dot{q}_i - L = \underline{\text{const}}$

New "Hamiltonian-like" formulation prob — sol'n continued

Apply "new G-formulation" to one-dimensional motion of mass m attached to a spring.



$$T = \tfrac{1}{2} m \dot{x}^2 \qquad V = \tfrac{1}{2} k x^2$$

$$L(x,\dot{x},t) = T - V = \tfrac{1}{2} m \dot{x}^2 - \tfrac{1}{2} k x^2$$

Now $\quad \dfrac{\partial L}{\partial \dot{x}} \equiv p_x = m\dot{x} \quad$ and $\quad \dfrac{\partial L}{\partial x} = \dot{p}_x = -kx$

Thus

$$G(p,\dot{p},t) = \dot{p}_x x + p_x \dot{x} - \tfrac{1}{2} m \dot{x}^2 + \tfrac{1}{2} k x^2$$

or

$$= -\frac{\dot{p}_x^2}{k} + \frac{p_x^2}{m} - \frac{1}{2} m \frac{p_x^2}{m^2} + \frac{1}{2} k \frac{\dot{p}_x^2}{k^2}$$

or

$$\boxed{G(p,\dot{p},t) = \frac{1}{2m} p_x^2 - \frac{1}{2k} \dot{p}_x^2}$$

∴ Eqs of motion:

$$\left.\begin{aligned} \frac{\partial G}{\partial p_x} = \dot{x} &\Rightarrow \frac{p_x}{m} = \dot{x} \Rightarrow m\dot{x} = p_x \\ \frac{\partial G}{\partial \dot{p}_x} = x &\Rightarrow -\frac{1}{k}\dot{p}_x = x \Rightarrow \dot{p}_x = -kx \end{aligned}\right\} \underbrace{m\ddot{x} = -kx}_{\text{S.H.M.}}$$

Also, $\dfrac{\partial G}{\partial t} = 0 \Rightarrow \dfrac{\partial L}{\partial t} = 0 \Rightarrow h = p_x \dot{x} - L = \text{constant}$.

or $\quad m\dot{x}^2 - \tfrac{1}{2} m\dot{x}^2 + \tfrac{1}{2} k x^2 = \tfrac{1}{2} m \dot{x}^2 + \tfrac{1}{2} k x^2$

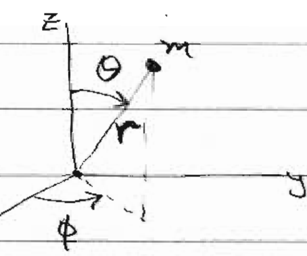
$= T + V = \text{constant}$

∴ Gives known correct result.

Hamiltonian formulation of m in central potential problem — sol'n

Use spherical polar coords (r, θ, ϕ)

(a) Find $H(r, \theta, \phi, p_r, p_\theta, p_\phi)$.



Must find L first.

$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2 \right)$$

$$V = V(r)$$

$$\therefore \ L = T - V = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2) - V(r)$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad \Rightarrow \dot{r} = p_r/m$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta} \quad \Rightarrow \dot{\theta} = p_\theta / m r^2$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m r^2 \sin^2\theta \, \dot{\phi} \Rightarrow \dot{\phi} = p_\phi / m r^2 \sin^2\phi$$

$$H = p_r \dot{r} + p_\theta \dot{\theta} + p_\phi \dot{\phi} - \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2) + V(r)$$

$$\boxed{H = \frac{1}{2m}\left(p_r^2 + \frac{1}{r^2} p_\theta^2 + \frac{1}{r^2 \sin^2\theta} p_\phi^2 \right) + V(r)}$$

(b) Since H does not depend explicitly on ϕ, $\frac{\partial H}{\partial \phi} = -\dot{p}_\phi = 0 \Rightarrow p_\phi = \text{const.}$

Since H does not dep. explicitly on t, H = const. of motion.

(c) Hamilton's eqs. of motion:

① $\frac{\partial H}{\partial p_r} = \dot{r} \Rightarrow m\dot{r} = p_r$

② $\frac{\partial H}{\partial p_\theta} = \dot{\theta} \Rightarrow mr^2\dot{\theta} = p_\theta$

③ $\frac{\partial H}{\partial p_\phi} = \dot{\phi} \Rightarrow mr^2\sin^2\theta\,\dot{\phi} = p_\phi$

④ $\frac{\partial H}{\partial \phi} = -\dot{p}_\phi \Rightarrow \dot{p}_\phi = 0 \Rightarrow p_\phi = \text{const.} = L_z$

⑤ $\frac{\partial H}{\partial \theta} = -\dot{p}_\theta \Rightarrow -\frac{1}{mr^2}\cdot\frac{\cos\theta}{\sin^3\theta}p_\phi^2 = -\dot{p}_\theta$

⑥ $\frac{\partial H}{\partial r} = -\dot{p}_r \Rightarrow -\frac{1}{m}\left(\frac{1}{r^3}p_\theta^2 + \frac{1}{r^3\sin^2\theta}p_\phi^2\right) + \frac{dV}{dr} = -\dot{p}_r \underset{\uparrow}{=} -m\ddot{r}$

using eq. ①

(d) Show the above eqs. lead to Lagrange's Eqs.

Lagrange Eq. for r: $\frac{\partial L}{\partial r} = mr\dot{\theta}^2 + mr\sin^2\theta\,\dot{\phi}^2 - \frac{dV}{dr}$

$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\ddot{r}$

$\left.\right\} m\ddot{r} = mr\dot{\theta}^2 + mr\sin^2\theta\,\dot{\phi}^2 - \frac{dV}{dr}$

Using eqs. ①, ②, & ③, to eliminate p_r, p_θ, & p_ϕ eq. ⑥ above becomes

$-m\ddot{r} = -mr^2\dot{\theta}^2 - mr\sin^2\theta\,\dot{\phi}^2 + \frac{dV}{dr}$ which is the Lagrange eq. for r.

Lagrange Eq. for θ: $\frac{\partial L}{\partial \theta} = mr^2\sin\theta\cos\theta\,\dot{\phi}^2$

$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta};\ \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta}$

$\left.\right\} mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = mr^2\sin\theta\cos\theta\,\dot{\phi}^2$

Using eqs. ②, ③ & ④, Eq. 5 becomes $-mr^2\sin\theta\cos\theta\,\dot{\phi}^2 = -mr^2\ddot{\theta} - 2mr\dot{r}\dot{\theta}$

Lagrange Eq. for φ: $\frac{dL}{\partial \phi} = 0 \Rightarrow \frac{\partial L}{\partial \dot{\phi}} = \text{const.} = mr^2\sin^2\theta\,\dot{\phi} = \text{const.} = \text{eq. ③ above}$

Done.

"Does a Hamiltonian exist for these eqns of motion" problem:

$$\dot{q} = p \sin q + q \cos p$$
$$\dot{p} = -(p \cos q + \sin p)$$

Does a function $H(q,p)$ exist that gives these eqns of motion?

If such an $H(q,p)$ exists then it must be differentiable. Therefore

$$\frac{\partial^2 H}{\partial q \partial p} = \frac{\partial^2 H}{\partial p \partial q}$$

or

$$\frac{\partial \dot{q}}{\partial q} = -\frac{\partial \dot{p}}{\partial p}$$

But, using the given eqns of motion

$$\frac{\partial \dot{q}}{\partial q} = p \cos q + \cos p$$

and $-\frac{\partial \dot{p}}{\partial p} = \cos q + \cos p$

we see that

$$\frac{\partial^2 H}{\partial q \partial p} = \frac{\partial^2 H}{\partial p \partial q} \Rightarrow p \cos q + \cancel{\cos p} = \cos q + \cancel{\cos p}$$

or $p \cos q = \cos q$.

But this will only be true if $\cos q = 0$ or $p = 1$. This severely limits p & q and is inconsistent with the eqns. of motion given.