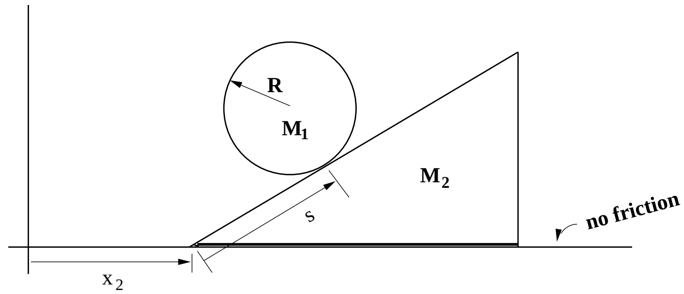
Phys 605. Midterm Exam

October 15, 2007

Problem 1: [45 pts.]A cylinder of radius R, mass M_1, and moment of inertia (about its central axis) $I = \frac{1}{2}M_1R^2$ is rolling without slipping on an incline with angle α with respect to the horizontal. The incline itself has mass M_2 and is free to slide without friction on a horizontal surface. Use s and x_2 as generalized coordinates as shown in the figure below.



(a) [10 pts] Clearly show that the Lagrangian is given by:

$$L = \frac{1}{2}(M_1 + M_2)\dot{x_2}^2 + \frac{3}{4}M_1\dot{s}^2 + M_1(\cos\alpha)\dot{s}\dot{x_2} - M_1 g(\sin\alpha)s$$

(b) [8 pts] Find any constants of motion and indicate the physical quantity each constant represents.
(c) [8 pts] Find the differential equations of motion for this system. (You do not need to solve these equations!)
(d) [7 pts] Now suppose a new constraint is added that forces x_2 to vary with time as: $x_2(t) = A\sin(\omega t)$, where A and ω are given constants. Find the new Lagrangian.
(e) [7 pts] Using the new Lagrangian, find any constants of motion and the new equation(s) of motion.
(f) [5 pts] Go back to part (d) and use a Lagrange multiplier, λ, to add the new constraint so as to obtain the constraint force associated with the new constraint. (Just set up the equations, you do not need to solve them.)

Problem 2: [25 pts.] A general surface of revolution may be described in cylindrical coordinates (r,ϕ,z) by the function $r = r(z)$. The function $r(z)$ and its derivative $dr/dz \equiv r'(z)$ are given. We want to find the equation for the curve that is the shortest path between two points on this surface.
(a) [8 pts] Show clearly that the expression for the differential path length on the surface that is the result of displacement dz and $d\phi$ is given by $(ds)^2 = (1 + r'^2)(dz)^2 + r^2(d\phi)^2$.
(b) [10 pts] Take z as the independent variable and show that the curve, $\phi(z)$, that is the shortest path between two points on the surface is given by

$$\phi(z) = \phi_0 + k\int_{z_0}^{z} \frac{\sqrt{r'^2(z') + 1}}{r(z')\sqrt{r^2(z') - k^2}}\, dz'.$$

(c) [7 pts] If the surface is a cylinder ($r = a$), show that the result given in part (b) is the equation for a helix.

Problem 3: [30 pts.] This problem involves launching a satellite of mass m into orbit about a spherical planet (with no atmosphere) of mass $M >> m$ and radius ρ. (Assume the mass of the planet is symmetrically distributed such that the potential $V(r) = -k/r$ applies for all $r > \rho$.) The mass m is raised to height $h = R - \rho$ (i.e. to radius R) and given a velocity $\boldsymbol{v}$ perpendicular to the radius. Use the results from the two-body problem with Kepler potential: $r(\theta) = C/(1 + \epsilon\cos\theta)$, where $C = \ell^2/\mu k$ and $E = \frac{\mu k^2}{2\ell^2}(\epsilon^2 - 1)$ to answer the following questions. Ignore the possibility of collision with the planet for the first two parts.
(a) [7 pts] Clearly show that the eccentricity ϵ as a function of $v \equiv |\boldsymbol{v}|$, is given by $\epsilon^2 = (\frac{Rv^2}{GM} - 1)^2$ where G is the gravitational constant in Newton's law of gravity.
(b) [8 pts] For what values of v is the orbit an ellipse, a circle, a parabola, and a hyperbola?
(c) [7 pts] Find the smallest v such that the satellite does not collide with the surface of the planet.
(d) [8 pts] Find the expression for ϵ if $\boldsymbol{v}$ is not perpendicular to the radius but at angle α to the perpendicular.
