H1.1 Average values and Root-Mean-Square Deviation (3P)

a) The function of displacement of the classical harmonic oscillator is given by 
   \( x(t) = A \sin(\omega t + \varphi) \). Calculate the average values 
   \[
   \overline{x^2} = \frac{T}{T} \int_0^T x^2(t) \, dt, \quad \text{and} \quad \overline{x} = \frac{T}{T} \int_0^T x(t) \, dt,
   \]
   and the corresponding values for \( \overline{p^2} \) and \( \overline{\bar{p}} \).

b) Calculate the root-mean-square deviation of \( x \) and \( p \), i.e.
   \[
   \Delta x = \sqrt{\overline{x^2} - \overline{x}^2} \quad \text{and} \quad \Delta p = \sqrt{\overline{p^2} - \overline{\bar{p}}^2},
   \]
   and the product \( \Delta x \cdot \Delta p \).

c) Using Bohr’s quantum condition
   \[
   \int_0^T p \, dx = \int_0^T p \dot{x} \, dt = n \hbar?
   \]
   find a relation for \( \Delta x \cdot \Delta p \) and discuss the meaning of this result.

H1.2 Matrix mechanics I (5P)

The Hamilton operator \( H \) of an harmonic oscillator is given by

\[
H = \frac{1}{2m} \overline{P^2} + \frac{1}{2} kX^2.
\]

In the Heisenberg formulation of quantum mechanics the operators \( P, X \) and \( H \) are expressed as matrices. \( X \) and \( P \) satisfy the commutation relation

\[
[X, P] = i \hbar \mathbf{1}.
\]

For simplicity we choose \( m = k = \hbar = 1 \), i.e.

\[
H = \frac{1}{2} \left( P^2 + X^2 \right) \quad [X, P] = i\mathbf{1}
\]

and define the matrix \( A = \frac{i}{\sqrt{2}} (P - iX) \) and its hermitian conjugate \( A^\dagger = \frac{1}{i\sqrt{2}} (P + iX) \).
Show that:

a) the matrices $A$ and $A^\dagger$ satisfy the relation $[A, A^\dagger] = 1$.

b) the matrix $H$ can be expressed as

$$H = \frac{1}{2} (AA^\dagger + A^\dagger A).$$

c) $[A, H] = A$ and $[A^\dagger, H] = -A^\dagger$

d) starting from the relation

$$[X, P] = c1 \quad (c \in \mathbb{C})$$

the dimension of the matrices $X$ and $P$ has to be infinite.

e) the infinite dimensional matrices

$$X = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & \sqrt{2} & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} & \cdots \\ 0 & 0 & 0 & \sqrt{4} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

and

$$P = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & -\sqrt{2} & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & -\sqrt{3} & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & -\sqrt{4} & \cdots \\ 0 & 0 & 0 & \sqrt{4} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

satisfy the relation $[X, P] = i1$.

f) Find the eigenvalues of the matrix $H = \frac{1}{2}(P^2 + X^2)$ using the above definitions of $X$ and $P$ and compare them with the result of the usual solutions of the quantum mechanical harmonic oscillator by functions.

**H8.2 Matrix mechanics II (2P)**

Let $A$ be the infinite dimensional matrix

$$A_{nm} = \begin{cases} \sqrt{n} & \text{for } n + 1 = m \\ 0 & \text{else} \end{cases} \quad n, m = 1, 2, 3, \cdots \infty,$$

and $N = A^\dagger A$.

a) Show: i) $[A, A^\dagger] = 1$ and ii) $[A, N] = A$

b) Calculate the elements of $N$ explicitly.